Intent matching

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This document is under construction. These are just some simple notes from ideas I am exploring regarding intent matching problems.

1 Theory

Let us consider a set of users sending swap orders with limit price.

When the swap order for the i-th user is executed, we have (S_i^e, B_i^e, Π_i^e), where S_i^e is the executed sell amount of token X, B_i^e is the executed buy amount of token Y, while Π_i^e is the executed price, defined as:

$$\Pi_i^e = \frac{S_i^e}{B_i^e} \leq \frac{S_i^{max}}{B_i^{min}} = \Pi_i^{lim} \tag{1}$$

We can think of each user's intent as a directed edge connecting two nodes, which are the X and Y tokens.

Thanks to this, finding matching intents, means finding closed loops in such intent-multigraph.

Given a closed loop $\mathcal{L}$ of N intents, we need to satisfy the following condition

$$\Pi_i^e \leq \Pi_i^{lim} \ , \forall i \in \mathcal{L} \tag{2}$$

Since in a closed loop $B_i^e = S_{i-1}^e$, then Eq. 2 reads also:

$$\frac{S_i^e}{S_{i-1}^e} = \Pi_i^{lim} \ , \forall i \in \mathcal{L} \ , \text{where if } i = 1 \text{ then } i - 1 = N \tag{3}$$

Thus, even if we are capable of finding a closed loop in the intent multigraph, we are not sure that this loop can be transformed into a valid set of transactions until we find at least a solution to Eq. 3 for all nodes in the loop $\mathcal{L}$.

Let us study the two limiting cases:

1.1 No Partial Fill

If no intent is partially fillable, the for each user $S_i^e \equiv S_i^{max}$. Thus, the solution to the problem exists only if:

$$\frac{S_i^{max}}{S_{i-1}^{max}} \leq \Pi_i^{lim} \ , \forall i \in \mathcal{L} \tag{4}$$

This condition can be checked in parallel for each i, and thus it is efficiently verifiable.

1.2 All Intents are Partially Fillable

If the intents will all admit partial fill, the situation is trickier.

Let's consider $\mathcal{L}$, a N-dimensional loop. A solution to the intent-matching problem exists if we can solve Eq. 3. This problem is self-consistent, i.e. S_i^e will depend on S_{i-1}^e. The minimal condition acceptable by each user is such that

$$\frac{S_i^e}{S_{i-1}^e} = \Pi_i^{lim} \ , \forall i \mathcal{L} \tag{5}$$

Let us now apply the following logarithmic transformation

$$log\left(\frac{S_i^e}{S_{i-1}^e}\right) = log\left(\Pi_i^{lim}\right) \tag{6}$$

By calling $log(A) = \tilde{a}$, and employing the properties of the log function, we can rewrite Eq. 6 as:

$$\tilde{s}_i^e - \tilde{s}_{i-1}^e = \tilde{\pi}_i^{lim} \tag{7}$$

which can be written in matrix form as:

$$\mathcal{M}\tilde{\mathbf{s}} = \tilde{\pi}^{lim} \tag{8}$$

Where

$$\tilde{\mathbf{a}} = [\tilde{a}_1, ..., \tilde{a}_N]^t \tag{9}$$

$$\mathcal{M} = \begin{bmatrix} 1 & 0 & \cdots & 0 & -1 \\ -1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -1 & 1 \end{bmatrix} \tag{10}$$

Thus $\mathcal{M}$ is a $N \times N$ matrix with ones on the diagonal and -1 in the lower sub-diagonal.

However,

$$det(\mathcal{M}) \equiv 0 \ , \forall N \tag{11}$$

This is the consequence of the fact that for $\mathcal{M}$:

$$\text{row}_1 = (-1) \cdot \sum_{i>1}^{N} \text{row}_i \tag{12}$$

Thus, our problem either has 0 ort ∞ solutions.
To evaluate if at least 1 solution (and thus ∞), exists, we can use the Rouche-Capelli theorem, i.e.

> Given $\mathbf{Ax} = \mathbf{b}$, if $\text{Rank}(\mathbf{A}) = \text{Rank}([\mathbf{A}|\mathbf{b}])$ then the system has ∞ solutions

This theorem can be used rather efficiently (e.g. Python).

1.2.1 Example: 2 partially fillable intents

Let us consider two users providing partially fillable intents, one selling X for Y and the other selling Y for X. Let us also assume that they agree on the relative price of tokens X and Y.

Then

$$\Pi_2^{lim} = \frac{1}{\Pi_1^{lim}} \tag{13}$$

Thus, our logarithmically-transformed problem reads:

$$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \tilde{s}_1 \\ \tilde{s_2} \end{bmatrix} = \begin{bmatrix} \tilde{\pi}_1 \\ -\tilde{\pi}_2 \end{bmatrix} \tag{14}$$

It is rather clear that

$$\text{Rank}\left(\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\right) = 1 = \text{Rank}\left(\begin{bmatrix} 1 & -1 & \tilde{\pi}_1 \\ -1 & 1 & \tilde{\pi}_2 \end{bmatrix}\right) \tag{15}$$

Thus, in this case, we have ∞ solutions. This makes sense since, if the two users agree on the relative price, then each coin exchange that preserves the price is a valid solution.